THE TEACHERS' SOLUTIONS TO P #1 TO P #7.

Problem (P #)	Levi	Mia

Problem (P #)

P #1: A farmer had 19 animals on his farm - some chickens and some cows. He also knew that there was a total of 62 legs on the animals on the farm. How many of each kind of animal did he have?

(Tripathi, 2008)

Levi

Solution 1: Algebraic

Let cows = A, chickens = B

We find two equations in the problem:

A + B = 19

And

4A + 2B = 62.

We rearrange the first to give B = 19-A, and then

We substitute this new equation into the second

equation for 'B', to get:

4A + 2(19 – A) = 62. Expanding and solving, we get:

4A + 38 – 2A = 62

2A + 38 = 62

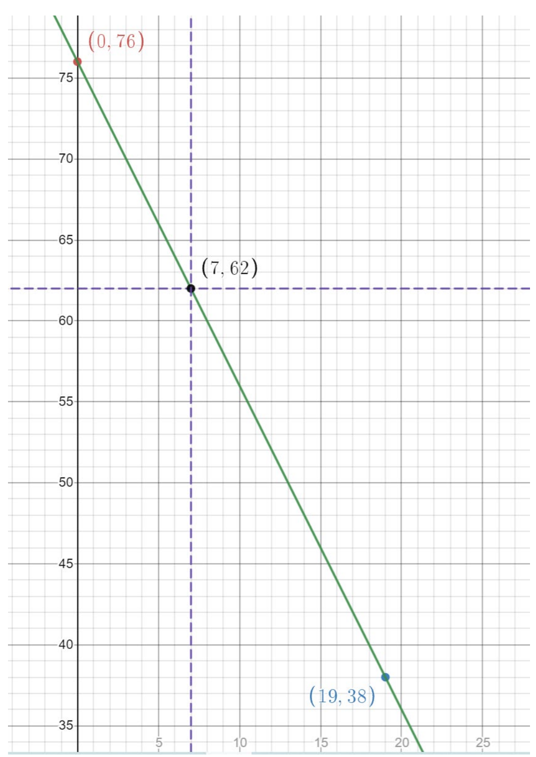
2A = 24

A = 12.

The farmer has 12 cows and 7 chickens.

Solution 2: Graphic (approximated with Desmos, but the method would be a "by-hand" method)

Make a graph with "legs" up the y-axis and "chickens" along the x-axis. We know that there are 19 animals. The number of chickens must be between 0 (where they're all cows and there is 76 legs) and 19 (all chickens, so 38 legs), so we plot those two points, as shown. We join the two points with a line, and then we go down the y-axis to find the number of chickens for when "legs" = 62. We go across to find out how many chickens that is (read it off the x-axis – it's 7). The farmer has 7 chickens, and so must also have 12 cows.



Mia

Solution A:

19 x 4 = 76 legs if they were all cows

76 -62 = 14 legs too many so subtract them from the original number of legs

62 – 14 = 48

48 ÷ 4 = 12 cows

7 chickens = 14 legs

Solution B: Guess and check

If there are 10 chickens x 2 legs that leaves 9 cows with 4 legs, which means 20 + 36 = 56 legs

Too low, so if there are 9 chickens and 10 cows there would be 9 x 2 + 10 x 4 = 18 + 40 = 58

Too low, so if there were 8 chickens and 11 cows 8 x 2 + 11 x 4 = 16 + 44 = 60

Too low, so if there were 7 chickens and 12 cows 7 x 2 + 12 x 4 = 14 + 48 = 62

Solution C:

Let chickens be a and cows be b

a + b = 19

2a + 4b = 62 therefore if we divide both sides by 2 to simplify the equation, we get a + 2b = 31

If we subtract the first equation from the second and solve; a + 2b – a – b = 31 – 19

Then b = 12 so a = 7

Solution D:

Chickens	Chicken legs	Cows	Cow legs	Total legs
0	0	19	76	76
1	2	18	72	74
2	4	17	68	72
3	6	16	64	70
4	8	15	60	68
5	10	14	56	66
6	12	13	52	64
7	14	12	48	62

Problem (P #)	**Levi**	**Mia**

P #2: Die A and Die B are twelve sides each. Suppose that you roll die A and die B at the same time. When do the dice satisfy the following two conditions?

(a) The sum of 2 times A plus B equals 15.
(b) 3 times A minus B equals 5.
(See Ito-Hino, 1995)

Levi

Solution A: List the possible outcomes

Let's think about condition (a), and what A can be. For example, A could roll 1, so 2 times A is 2, but then B would need to be 13 to add up to 15. The dice don't have the number 13, so this won't work. A cannot be 1.

A	2A	B
2	4	11
3	6	9
4	8	7
5	10	5
6	12	3
7	14	1

Now, by inspection, do any of these arrangements meet the second condition? I'll try each of the A values to see if any of them work. I'll use a table to help me out.

A	3A-5	B
2	1	11
3	4	9
4	**7**	**7**
5	10	5
6	13	3
7	16	1

I see that when A = 4, B = 7, we have a solution.

Solution B:

Condition 1: 2A + B = 15

Condition 2: **3A – 5 = B**

By substituting Condition 2 into Condition 1, we get

2A + **3A – 5** = 15, and so 5A – 5 = 15.

5A = 20, and so A = 4 is the correct solution.

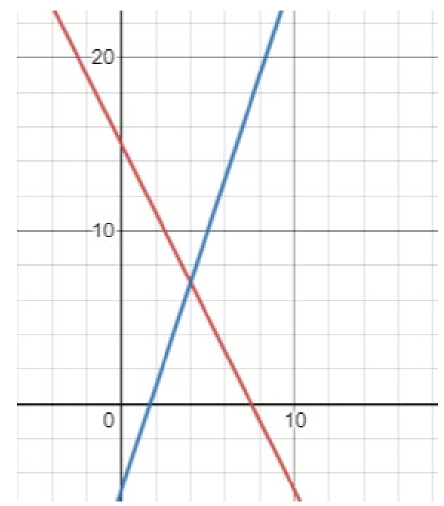
Mia

Solution A

(a) 2a + b = 15
(b) 3a – b = 5

A	B	Does this work with second equation?
2	11	No
3	9	no
4	7	yes
5	5	No

Solution B



Solution C
Solve for b
2a + b = 15
b = 15 – 2a
Substitute in to second equation
3a – b = 5
3a – (15-2a) = 5
5a -15 = 5
5a = 20
a = 4
Substitute a back into equation to find b
3 x 4 – b = 5
12 – b = 5
12 – 5 = b
7 = b

Solution D
Guess and check
Similar to the table but more random.

Problem (P #)	Levi	Mia

Problem (P #)

P #3: You have some teen and young adult books. You gave one-half of the books plus one to a friend, one-half of the remaining books plus one to another friend, and one-half of the remaining books plus one to another friend. If you have one book left for you, how many books did you have at the start?

(Adapted from Musser et al., 2008)

Levi

Solution one: A visual method that starts from the end.

I draw my book with a black "**X**". Then I draw enough books that went to the friend in a different colour, "**X**".

"I give half the remaining books plus one more to a friend, and at the finish I have one".

I draw my book with a black X. I add the "plus one more" book with a red X. Then I add in that many X's again. That is, a black X plus one red X is 2, so I add two more red X's.

X X   X X

Ok, let's extend that method. Clearly before that step I had four, so let's add four black X's and one red X. That'll make five, so I'll add in five more black X's. That makes ten in total.

X X X X X   X X X X X

Finally, we're up to the first step. Clearly this had left me with ten books. I draw my ten, I add one more which makes 11, so I add on 11 red X's.

X X X X X X X X X X X   X X X X X X X X X X X

I see that I must have started with 22 books. I then gave away 11 + 1 (leaving me with 10). Then I gave away 5 + 1 (leaving me with 4). Then I gave away 2 + 1, leaving me with one remaining book. 22 is the correct answer.

Solution Two: Algebra.

If I have "A" books now, I had "B" books before. A is half of B, minus one more, so

$$A = \frac{B}{2} - 1$$

To figure out B, I can add the one back on, and then double it, so if I know A I can get B by

$$2(A + 1) = B$$

A	2(A+1) = B
1	2*(**1**+1) = 4
4	2*(**4**+1) = 10
10	2*(**10**+1) = **22**

This shows that I had 22 books before the first step.

Mia

Solution A Working backwards

Ending with 1

1 Plus 1 = ½ of the books

4 books

4 books plus 1 = ½ of the books

10 books

10 books plus 1 = ½ of the books

22 books

Check working forwards

Half of 22 is 11, plus 1 makes 12, leaving me with 10

Half of 10 is 5, plus 1 makes 6, leaving me with 4

Half of 4 is 2, plus 1 makes 3, leaving me with 1

Solution B

b ÷ 2 -1 = 1

b ÷ 2 = 2

b = 4

Using this formula: b = x/2 – 1

4 = x/2 – 1

5 = x/2

10 = x, use this for my new number of books (b)

10 = x/2 – 1

11 = x/2

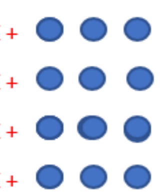
X = 22

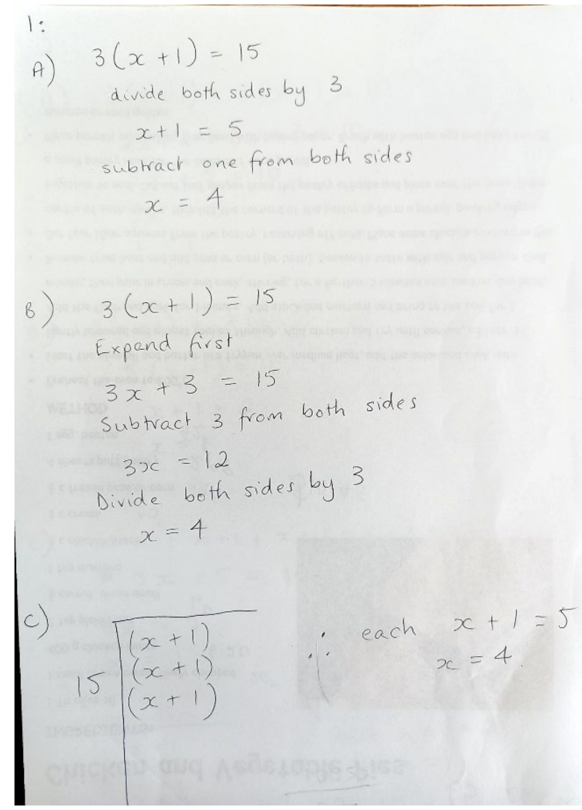
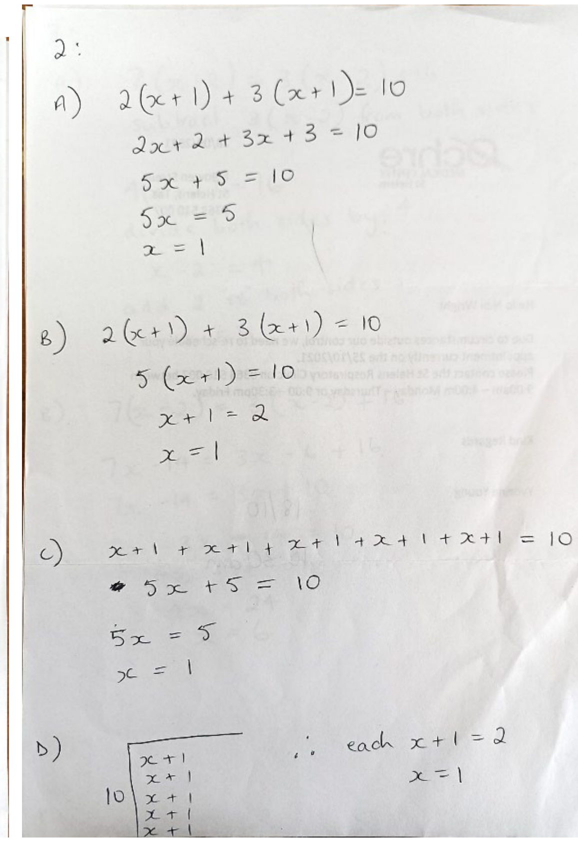
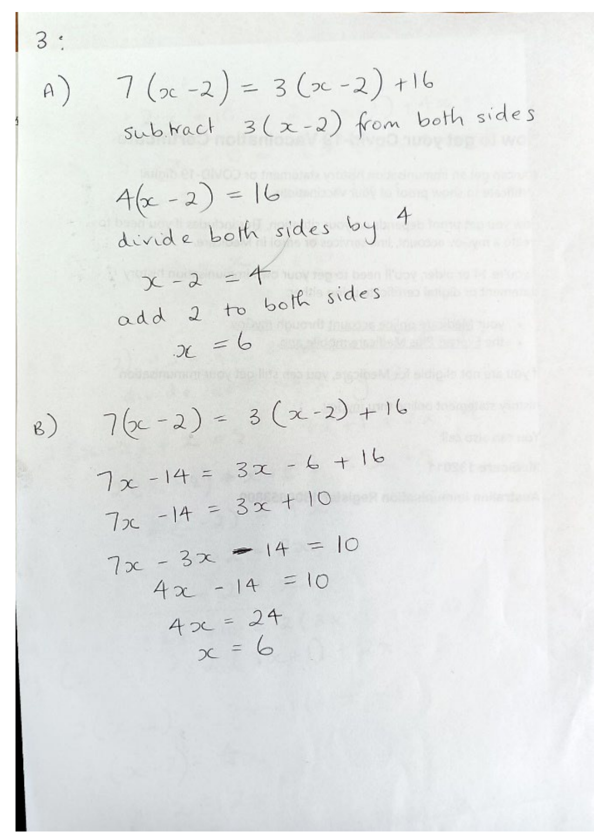
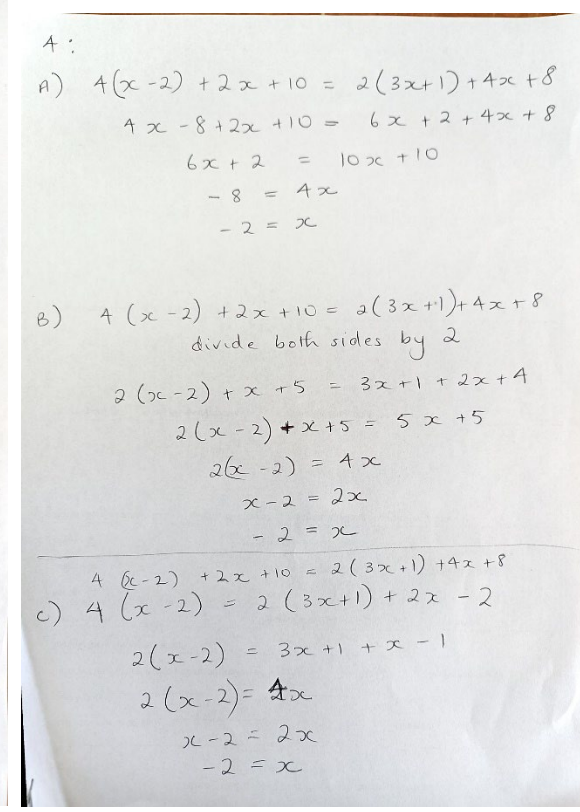
Solution C

Act it out with books.

Solution D Guess and Check

Books	Half + 1	Leftover
40	21	19
19	Not whole number	
30	16	14
14	8	6
6	4	2 too high
20	11	9
9	Not whole number	
24	13	11
11	Not whole number	
22	**12**	**10**
10	**6**	**4**
4	**3**	**1**

Problem (P #)	Levi	Mia
P #4: Solve the equations below for *x*: (a) $4 \times (x+3) = 16x$ (b) $2.\left(\frac{3(2n-1)}{7}+6\right)+7=25$ (Star & Seifert, 2006)	**P #4a** Solution A: Opposite Functions: $4(x+3) = 16x$ $x+3=4x$ $3=3x$ Therefore x = 1. Also $4(x+3)=16x$ $4x+12=16x$ $12=12x$ Therefore x = 1 **P #4b** $2\left(\frac{3(n-1)}{7}+6\right)+7=25$ $2\left(\frac{3(n-1)}{7}+6\right)=18$ $\frac{3(n-1)}{7}+6=9$ $\frac{3(n-1)}{7}=3$ $\frac{(n-1)}{7}=1$ $n-1=7$ Therefore n = 8	**P #4a** Solution A 1. $4\,(x\ +\ 3)\ =\ 16x$ Divide both sides by 4 $x+3=4x$ subtract x from both sides $3=3x$ divide both sides by 3 $x=1$ substitute back in to equation to check $4\quad(1+3)=16$ Solution B $4\,(x\ +\ 3)\ =\ 16x$ Expand brackets $4x\ +\ 12\ =\ 16x$ Subtract 4x from both sides $12\ =\ 12\,x$ Divide both sides by 12 $1\ =\ x$ Solution C Think 4 times what equals 16? 4 so the brackets must equal 4 Therefore x = 1 Substitute to check Solution D Draw images of x's and apples X + X + X + X +  = X+X+X+X+X+X+X+X+X+X+X+X+X+X+X+X Cancel out four of the x's from each side  = X+X+X+X+X+X+X+X+X+X+X+X Therefore, each X = 1 apple **P #4b** Equation 2 $\left(\frac{3(2n-1)}{7}+6\right)+7=25$ Working from the outside/backtracking whilst keeping both sides balanced Subtract 7 from both sides $\left(\frac{3(2n-1)}{7}+6\right)=18$ Subtract 6 from both sides $\left(\frac{3(2n-1)}{7}\right)=12$ Multiply both sides by 7 $3(2n-1)\ =\ 84$ Divide both sides by 3 $2n-1=\ 28$ Add 1 to both sides $2n\ =\ 29$ Divide both sides by 2 $n\ =14.5$

Problem (P #)	Levi	Mia
P #5: Solve the equations below for *x*: (a) $3(x+1) = 15$ (b) $2(x+1) + 3(x+1) = 10$ (c) $7(x-2) = 3(x-2) + 16$ (d) $4(x-2) + 2x + 10 = 2(3x+1) + 4x + 8$ (Star & Seifert, 2006)	**P #5a** Start by expanding. 3x + 3 = 15. Subtract 3 from both sides. 3x = 12 Therefore x = 4 (by inspection) **P #5b** Start by expanding 2x + 2 + 3x + 3 = 10 Then collect like terms. 5x + 5 = 10 Subtract 5 from both sides 5x = 5 Therefore x = 1 (by inspection) —— Another method is to add together the "like terms" brackets. 5(x+1) = 10 Divide both sides by 5 X + 1 = 2 Therefore x = 1 **P #5c** Start by expanding 7x – 14 = 3x – 6 + 16 7x – 14 = 3x + 10 Add 14, subtract 3x 4x = 24 Therefore x = 6. —— Another method is to collect "like terms" brackets. Subtracting 3(x-2) from both sides, 4(x-2) = 16 Dividing by 4, X – 2 = 4 Therefore x = 6. **P #5d** Start by expanding 4x – 8 + 2x + 10 = 6x + 2 + 4x + 8 6x + 2 = 10x + 10 Rearrange to give -4x = 8 X = -2	   

Problem (P #)	Levi	Mia

Problem (P #)

P #6: If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are.

(See Kieran, 1992)

Levi

Solution 1 – making a list:

Suppose we knew the numbers summed to 15. Let's think about the possible solutions (sticking for now with the natural numbers)

No's	1+14	2+13	3+12	4+11	5+10	6+9	7+8
Diff	13	11	9	7	5	3	1

We see there's no double ups. This means that if you know the sum and you know the difference then you can just list out all the numbers that add up to that amount and find the ones with the required difference. This method could also extend to problems where sum and difference are any numbers – not just counting numbers but integers and non-integers too.

Solution 2 – algebra and substitution – a general solution.

Let a be larger than b.

Let a + b = X (the sum)

Let a – b = Y (the difference)

We can rearrange the second equation to get a = b + Y

We can substitute that into the first to find b + b + Y = X

So 2b + Y = X.

It follows that if we are given any Y and any X, we can substitute them into this equation to find one of the numbers, and then find the other one. **For example,** if they summed to 15 with a difference of 3, then X = 15, Y = 3, and 2b + 3 = 15, so 2b = 12, and then b = 6.

Mia

Solution A

$x + y = 14$ $x - y = 2$ combine 2 equations

$$2x = 16$$

$$x = 8$$

$therefore\ y = 6$

Solution B

$$x + y = 14$$
$$x - y = 2$$
$$x + y = 14\ rearrange\ second\ equation\ x = y + 2$$
$$substitute\ for\ x$$

$y + 2 + y = 14$

$2y + 2 = 14$

$y = 12$

y=6 x=8

Solution C

Guess and check/list/logic

$x + y = 14$ $x - y = 2$

X must be bigger than y

7 +7 =14   7-7=0

8 + 6 = 14   8 – 6 = 2

Problem (P #)	Levi	Mia
P #7a: Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it? ***P #7b***: A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Using algebra, show that she is right. ***P #7c***: Show, using algebra, that the sum of two consecutive numbers is always an odd number. (See Kieran, 1992)	**P #7a:** First, a few goes with numbers. a. Let the numbers be 4, 5, 6. 5^2 = 25, and 4*6=24. 25 – 24 = 1. b. Let the numbers be 9, 10, 11. 10^2 = 100 and 9*11 = 99. 100 – 99 = 1. I suppose the difference must always be 1. Now, with algebra: c. Let the numbers be n – 1, n, and n + 1. (n-1) . (n+1) = $n^2 – 1$ (this is a difference of two squares relationship). if the square of the middle term is n^2 and the product of the outer terms is n^2-1 then we see why the product is always one less than the square of the middle term. **P #7b:** $\frac{5x+12-x}{4} = \frac{4x+12}{4} = x + 3$. If her starting number is 'x', her answer will be 'x+3', proving she is right. **P #7c:** "2n" gives us the sequence of even numbers and "2n+1" gives us the sequence of odd numbers. Let the lower number be 'n' and the upper number be 'n+1'. Clearly if we add them together, we get '2n+1', which by definition is an odd number.	**P #7a:** 9, 10, 11 100 – 99 = 1 3, 4, 5 16 – 15 = 1 x^2 - (x – 1) (x + 1) =x^2 - (x^2 +1x - 1x – 1) =1 Or (x-1) (x+1) = x^2 + x – x – 1 = x^2 - 1 **P #7b:** **Solution A:** $\frac{5n+12-n}{4}$ = $\frac{4n+12}{4}$ = $n + 3$ Solution B: (5n + 12 – n) ÷ 4 = (4n + 12) ÷ 4 (divide both numbers by 4) = n + 3 **P #7c:** $x + x + 1$ e.g., 5 + 6 = 5 + 5 + 1 This will always be double a number + 1