European Journal of Science and Mathematics Education, 2025, **13**(1), 41-57
ISSN: 2301-251X (Online)

OPEN ACCESS

Research Article

Mathematical practices (commonly used by mathematicians) in mathematics teachers' solutions to algebraic problems

Vesife Hatisaru [1,2*]
0000-0003-2101-1764

Steven Richardson [1]
0000-0002-5723-7346

Jon R. Star [3]
0000-0002-4830-3815

[1] Edith Cowan University, Joondalup, AUSTRALIA
[2] University of Tasmania, Hobart, AUSTRALIA
[3] Harvard Graduate School of Education, Cambridge, MA, USA
[*] Corresponding author: v.hatisaru@ecu.edu.au

Citation: Hatisaru, V., Richardson, S., & Star, J. R. (2025). Mathematical practices (commonly used by mathematicians) in mathematics teachers' solutions to algebraic problems. *European Journal of Science and Mathematics Education, 13*(1), 41-57. https://doi.org/10.30935/scimath/15889

ARTICLE INFO

Received: 4 Aug 2024

Accepted: 12 Dec 2024

ABSTRACT

A teacher of mathematics knows mathematics as a teacher and as a mathematician. Whilst the existing research on teacher knowledge contributes to our understanding of the ways of knowing mathematics as a teacher, little is known about ways of knowing mathematics as a mathematician. Guided by the conceptual framework of mathematical practices (MPs) (commonly used by mathematicians), this case study aimed to contribute to fill this gap. The study examined solutions generated by two secondary teachers of mathematics to a set of algebraic problems to determine which MPs are apparent, or not, in the teachers' work. Data were content analyzed deductively. Findings reveal that both teachers consistently demonstrated three practices: *seeking to find patterns*; *creating models for mathematical ideas*; and *using symbolic representations of ideas*, whilst two practices: *using precise definitions of objects*; and *having fine distinctions about language* were less present in either teacher solutions. More high-level practices such as *characterizing objects based on structure* and *using logical arguments as sources of conviction* were manifested in routine problems but absent in nonroutine problems. It is anticipated that teacher training experiences and curriculum contexts have influences on teachers' MPs in doing mathematics.

Keywords: algebraic problems, mathematical practices, mathematicians, mathematics teachers, secondary mathematics

INTRODUCTION

For a teacher, there are at least three main ways of knowing mathematics.

(1) *A teacher knows mathematics as a scholar*. That is, teachers know about mainstream mathematics, including major results, the history of the discipline's ideas, and the discipline's connections to precollege mathematics.

(2) *A teacher knows mathematics as a teacher*. Teachers use mathematics in ways that are distinct from teaching, including planning, task design, and interpreting student thoughts. They also understand the thinking that underlies different branches of mathematics, and how this thinking can be developed in students.

(3) *A teacher knows mathematics as a mathematician*. They often immerse themselves in mathematics that includes grappling with problems and carrying out experiments, building abstraction from the experiments, and constructing arguments that bring coherence to the abstraction (Matsuura et al., 2013).

Most teacher education and development programs–as also stated by Matsuura et al. (2013)–focus primarily on the first two ways of knowing mathematics. While the existing research on teacher mathematical knowledge contributes to our understanding of ways of knowing mathematics as a teacher (i.e., *#2 above*; Hatisaru & Erbaş, 2017; Hill & Chin, 2018) less is known about *#3*: ways of knowing mathematics as a mathematician. In this study, we aim to contribute to fill this gap by investigating how secondary mathematics teachers use the ways of doing mathematics that are usually used by mathematicians. We employ the notion of mathematical practices (MPs) to conceptualize the study.

THE CONCEPT OF MATHEMATICAL PRACTICES

There is not a common conceptualization of MPs (Moschkovich, 2013), and different terms are used in the research to define and describe these practices such as mathematical habits of mind (Levasseur & Cuoco, 2003), mathematical sophistication (Seaman & Szydlik, 2007), mathematical proficiency (Kilpatrick et al., 2001), mathematical competencies (Niss & Højgaard, 2019), or mathematical practices (National Council of Teachers of Mathematics [NCTM], 2014). It is, however, consistent within the relevant literature that MPs are about how mathematical work is done (Cuoco et al., 2010), that they are applied "… across all content areas of mathematics rather than to an understanding of a specific definition, mathematical object, or procedure" (Seaman & Szydlik, 2007, p. 172), and that they are implicated well where individuals undertake mathematical tasks (Tran & Munro, 2019).

To understand the ways that mathematics teachers use MPs in doing mathematics, we accept the learning and/or doing of mathematics as both a cognitive and a social process. To us, mathematics teachers understand and use MPs as a way of acting, thinking, and talking with mathematics, and commonly in social settings–for example in a classroom (Moschkovich, 2013). We believe that depending on the community of thought teachers are exposed to, teachers' experiences can impact their view of mathematics and what it means to have mathematical experiences. Ranging from merely looking for a solution with little systematic approaches to investigating, connecting, testing, and seeking to understand, experiences that mold one's level of mathematical sophistication. The more mathematical experiences teachers are provided, the higher the likelihood that they learn to experience mathematics in a more sophisticated way (Bauer & Kuennen, 2016; Seaman & Szydlik, 2007).

We employ Seaman and Szydlik's (2007) set of MPs (see **Table 1**). In proposing them, the authors conducted a comprehensive literature review into the practices of mathematicians when they create mathematics, particularly their habits and values. These practices were empirically measured by Szydlik et al. (2009).

Prior Research on Teachers' Use of Mathematical Practices

Existing research on teachers' use of MPs is comparatively sparse, but by looking across the various terminological conceptions of MPs (e.g., mathematical sophistication, mathematical habits of mind, mathematical proficiency), it is possible to get a picture of the existing research base on this topic. A few studies examined MPs through the lens and terminology of mathematical sophistication. In particular, Seaman and Szydlik (2007) examined eleven preservice elementary teachers' mathematical sophistication in an attempt to explain the reason why these pre-service teachers (PSTs) were unable to make sense of mathematics. It was found that the participants "… did not attend carefully to language in a story problem, and they did not attempt to use relevant explanations" (p. 180). According to the authors, both mathematical knowledge for teaching and mathematical sophistication are difficult to attain, and therefore, further work is needed on increasing mathematical sophistication among PSTs in both mathematics content and mathematics education coursework.

As other examples, Lim (2008) aimed to advance PSTs' mathematical sophistication through some mathematical tasks that the author designed, but the study contained no empirical data. More empirically,

Table 1. MPs identified by Seaman and Szydlik (2007)

MP	Description
MP1: Seek to find and understand patterns.	Value and use patterns and regularity
	Have systematic ways of making sense of patterns involving number and shape
MP2: Classify and characterize objects based on structure.	Value/use operational or geometric properties over mathematically superficial ones such as orientation, problem context, or labelling
MP3: Make and test conjectures about objects and structures.	Explore problem situations
	Make and test conjectures by considering extreme or divergent cases
MP4: Create mental and/or physical models for, and examples and non-examples of, mathematical objects.	Draw or imagine models (general or dynamic) to make sense of problem situations, relationships, and novel definitions
MP5: Value and use precise definitions of objects.	Use the mathematical definition to classify objects without regard to extraneous meanings of terms suggested by the wider culture
MP6: Value an understanding of why relationships make sense.	Recognize that mathematics makes sense
	Seize opportunities to explore relationships
MP7: Value and use logical arguments and counterexamples as sources of conviction.	Understand that examples alone do not provide sufficient mathematical justification for a claim
	At the same time recognize that an example can provide the seed of a general argument
	Value counterexamples and arguments based on structure and reasoning
MP8: Value precise language and have fine distinctions about language.	Understand and use the mathematical culture's normative meanings for terms such as 'and' and 'or'
	Distinguish necessary from sufficient conditions
	Distinguish converse from contrapositive forms
	e.g., if a person has ten pets, it is also true that they have two pets
MP9: Value and use symbolic representations of, and notation for, objects and ideas.	Understand and use the mathematically normative meanings for familiar symbols persevere to make sense of a new symbol or a new notation that is defined for them

Bauer and Kuennen (2016) investigated to what extend the instrument constructed by Szydlik et al. (2009), based on the MPs in **Table 1**, was suitable for use with German secondary (gymnasium) PSTs, and the differences between novice and expert PSTs in terms of the level of mathematical sophistication they had. The instrument (which was designed for use with elementary PSTs) worked well with German secondary PSTs, perhaps because, as the authors stated, the instrument, and identified MPs characterizing it, is not bound to specific mathematical content. Within a subset of the study sample, expert PSTs (final years) were found to be more sophisticated than novice PSTs (beginning years). This finding indicates that teacher mathematical sophistication may look different after teachers go into the profession. That is, practicing teachers' use of MPs in doing mathematical work–which is the focus of this study–may differ from those of PSTs.

Looking to research that framed MPs as habits of mind, Matsuura et al. (2013) studied secondary teachers' mathematical habits of mind within a research program whose goal was understanding the associations between secondary teachers' mathematical knowledge for teaching and their students' mathematical understanding. The authors reported three teachers' habit of using mathematical language (MP5 and MP8 in **Table 1**), how this habit was manifested in the classroom, and how it might influence student learning. Their study showed that depending on how the teachers used it, the habit of using precise mathematical language could either support or inhibit student understanding.

A study by Copur-Gencturk and Doleck (2021) is quite similar to the notion of MPs as described above. This study investigated teachers' strategic competence, defined as the ability to mathematise a problem, devise a valid solution strategy to solve it, and arrive at a correct answer. A total of 350 fourth- and fifth- grade teachers were surveyed to examine their strategic competence in the context of solving four multistep fraction word problems. Most teachers, in this study, who generated an appropriate strategy arrived at a correct answer, and many of the teachers who made errors in devising a valid strategy were unable to find a correct answer. According to the authors, the teachers' strategic competence was highly contingent on whether they devised a valid solution strategy to solve the given problems correctly. This finding indicates that teachers' performance in solving problems may be mediated by the ways that they use, or not, MPs, though investigating such associations was not within the scope of the current research.

It is worth noting that, although not widely investigated among teachers, MPs, particularly as conceptualized as mathematical habits of mind and strategic competence, has been investigated among school-aged learners (Cuocu et al., 2010). As these studies indicate, the lens of MPs has been quite informative for investigating students' mathematical thinking and problem-solving, and thus there is reason to anticipate that this framework will be similarly useful when applied to teachers.

THE CURRENT STUDY AND ITS SIGNIFICANCE

This study investigates how secondary mathematics teachers use the ways of doing mathematics that mathematicians usually use. By using the notion of MPs, we document these practices in two secondary mathematics teachers' solutions to a set of algebra problems. The research questions (RQs) we explore are, as follows:

RQ1: How do the teachers solve the given problems? What strategies do they generate?

RQ2: What mathematical practices are manifested in the teachers' solutions?

The study is significant for several reasons. Firstly, the mathematics education community at large considers mathematics disciplinary practices–including generalizing, symbolizing, structuring, representing, modelling, conjecturing, proving, seeking efficiency, and seeking elegance (Lim, 2013)–as fundamental aspects of knowing, doing, and teaching mathematics (Levasseur & Cuoco, 2003; Matsuura et al., 2013). Developing these practices in students is considered the essence of mathematics learning and teaching as they provide students with a repertoire of general strategies and methods that can be applied in a variety of situations (Turner et al., 2015). Consequently, MPs have increasingly become globally accepted outcomes of learning mathematics. They have influenced school curricula in different ways in many countries including in the United States (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010), Australia (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2022), Norway and Sweden (Boesen et al., 2018). Mathematical knowledge and goals of mathematics education have been partly described based on the extent to which these practices are acquired (Turner et al., 2015). They have also appeared in large scale international comparison studies such as the Programme for International Student Assessment (PISA) (Thomson et al., 2019). Our current investigation is important because MPs have utility in both primary and secondary mathematics classes and beyond (Glover, 2019).

Secondly, MPs are considered as a related facet of mathematical knowledge needed for teaching (Matsuura et al., 2013). As described by Ball et al. (2005): "Teachers must be able to do the mathematics they are teaching, but that is not sufficient knowledge for teaching … Fluency, accuracy, and precision in the use of mathematical terms and symbolic notation are also crucial" (p. 1058). Whilst there is abundance of research studies on mathematics teachers' (for example) mathematics content knowledge or mathematics pedagogical content knowledge (Tchoshanov et al., 2017), little is known about teacher capability in the use of MPs. If the goal of mathematics teaching is to have students develop these practices, or to bring the practices of mathematicians into the classroom (Moschkovich, 2013), then it is necessary to have insights into the ways that teachers use MPs.

Finally, the existing research studies have largely focused on mathematicians' use of MPs (e.g., Martín-Molina et al., 2018) or situated in postsecondary education in content domains including linear algebra, differential equations, or analysis (see Glover, 2019). Little is known about how MPs appear in pre-college classrooms (Moschkovich, 2013). In this study, we apply the MPs concept to mathematics teachers' work, and this will help mathematics educators and researchers in at least two ways. First, our analysis can aid in the development of theoretical clarity about mathematics practices as used by teachers. And second, this study provides the field with grounded examples that can help to develop clarity in conversations with practitioners around MPs.

METHODS

This is a case study wherein it was aimed to obtain an in-depth appreciation of a construct of interest (MPs) in its natural context (solutions given by teachers to mathematics problems) (Crowe et al., 2011). The study was grounded in a teacher study group, a professional learning (PL) project, implemented by the first author in Tasmania (Australia), wherein secondary mathematics teachers were supported to develop deeper understanding of algebra (Hatisaru, 2024). Teachers of mathematics who attended an annual state-based conference, organized by the Mathematical Association of Tasmania, were invited to this PL project. Six teachers expressed interest, while not all were able to attend all study group meetings. This paper focuses on MPs that were manifested in the mathematical work of two teachers–Levi and Mia (altered names)–who attended all PL meetings. Mia holds a bachelor of primary education that contained some elective units in year 7 to year 10 mathematics, and Levi has a bachelor of secondary education degree with a mathematics minor. As part of this degree, Levi undertook calculus and algebra units. At the time of the study, they were teaching in two different public secondary schools to year 7 to year 10 students (12–16 years old). Mia had three and Levi had six years of experience in mathematics teaching.

The PL lasted one school year, between September 2021 and March 2022. During this time, seven PL meetings were held, approximately once a month. All meetings were held virtually. Usually one week before each meeting, the author sent the teachers a reflection form with a problem recorded in it. The teachers' solutions to the problems guided the substance and direction of discussions at each meeting. As this was a voluntary PL opportunity for them, we assume that the solutions presented by the teachers belonged to them.

Data Source: Reflection Forms

Within the study group, multiple data collection instruments were administered, including an initial questionnaire which was designed to evaluate the teachers' mathematics teaching background, reflection forms–the focus of this paper, and a final questionnaire designed to have the teachers' reflections upon the ways in which the study group supported their PL (Hatisaru, 2024).

The reflection forms comprise two pages. On the first page, the teachers were given a problem and prompted to solve the problem in as many different possible ways as they could:

> Prompt #1: Think and explain as many different possible solutions to the problem as you can. Name the solutions as solution A, solution B, solution C, and so on.

It was aimed to gain insight into their capability to formulate, represent and solve mathematical problems. On the next page, the teachers were presented with three prompt questions that aimed to determine how familiar the problems were to them, the specific strategies that they would use in the teaching of the respective problem, and solution method(s) that they would desire from their students:

> Prompt #2: Would this problem be useful for the year levels you teach? Does it resemble problems you might use?
>
> Prompt #3: If so, identify the solution(s) (e.g., solution A, solution B, etc.) that you would use to solve it, and why?
>
> Prompt #4: Identify the solution(s) you hope your students would use.

The teachers were emailed the reflection forms with the relevant problem recorded on it (problems #1 to #7, see **Table 2**), usually one week before each of the study group meetings (meeting #1 to meeting #7). They used the forms to record their solutions and brought their solutions to share with the group.

The Problems Used

The problems (P #) used in this study are given in **Table 2**, grouped according to the mathematical content that they cover. P #1, P #2, and P #6 are essentially simultaneous equations problems that can be solved by standard methods including using a graph, drawing a pictorial model, or using an algebraic approach (Tripathi, 2008). All three problems involve translating verbal statements into symbolic equations with P #6 additionally

Table 2. Problems used in this study grouped according to their content

Content	Problem (P #)
Simultaneous linear equations	*P #1*: A farmer had 19 animals on his farm–Some chickens and some cows. He also knew that there was a total of 62 legs on the animals on the farm. How many of each kind of animal did he have? (Tripathi, 2008). *P #2*: Die A and die B have twelve sides each. Suppose that you roll die A and die B at the same time. When do the dice satisfy the following two conditions: (a) The sum of 2 times A plus B equals 15, and (b) 3 times A minus B equals 5? (Ito-Hino, 1995). *P #6*: If you are given the sum and difference of any two numbers, show that you can always find out what the numbers are (Kieran, 1992).
Linear equations	*P #3*: You have some teen and young adult books. You gave one-half of the books plus one to a friend, one-half of the remaining books plus one to another friend, and one-half of the remaining books plus one to another friend. If you had one book left for you, how many books did you have at the start? (Musser et al., 2008). *P #4 and P #5*: Solve the equations below for *x* (sample items): P #4a: $4 \times (x+3) = 16x$. P #4b: $2 \times \left(\frac{3(2n-1)}{7} + 6\right) + 7 = 25$. P #5b: $2(x+1) + 3(x+1) = 10$ (Star & Seifert, 2006).
Using letters to express the general	*P #7a*: Take three consecutive numbers. Now calculate the square of the middle one, subtract from it the product of the other two. Now do it with another three consecutive numbers. Can you explain it with numbers? Can you use algebra to explain it? *P #7b*: A girl multiplies a number by 5 and then adds 12. She then subtracts the original number and divides the result by 4. She notices that the answer she gets is 3 more than the number she started with. She says, "I think that would happen, whatever number I started with." Using algebra, show that she is right. *P #7c*: Show, using algebra, that the sum of two consecutive numbers is always an odd number (Kieran, 1992).

incorporating variable coefficients. That is, (for example) in P #1, the number of animals on the farm and the total number of legs are specified (i.e., 19 and 62, respectively); however, in P #6 the sum and difference are not specified, and therefore they must be represented as variables to maintain the necessary generality. In P #1, the aim is to solve for the unknown number of cows and chickens, while for P #6 the aim is to demonstrate that the two unknown numbers can be uniquely specified in terms of their sum and difference (see Hatisaru et al., 2022).

P #3, P #4, and P #5 are problems involving linear equations. P #3 describes an iterative process in which each iterative stage can be represented by a linear equation relating the number of books remaining after that stage to the number of books available at the beginning of the stage. Although neither teacher attempted to do so, it is possible to derive an expression to calculate the initial number of books using a single equation. If y denotes the initial number of books, and x denotes the number of books remaining after n stages, then $y = x \times 2^n + 2 \times (2^n - 1)$. The current scenario has 3 stages ($n = 3$) in which books are distributed, and the remaining number of books $x = 1$. Hence $y = 1 \times 2^3 + 2 \times (2^3 - 1) = 8 + 14 = 22$. P #4 and P #5 are distinct from P #3 as they do not involve the formulation of an equation from a worded problem. Both include routine, abstract equations that can be solved by using the rules for manipulating algebraic symbols (e.g., expanding, collecting like terms, etc.).

P #7 has three items, and while all target an algebraic solution each item has nuances. Specifically, P #7a instructs the person answering the question to trial specific numbers. P #7b (also P #7c) asks for algebra to be used to show the result; but P #7b tells the person what they should observe, while P #7a does not.

These problems were selected such that all three types of school algebra activities would be covered: representational (P #1, P #2, and P #6); rule-based (P #3, P #4, and P #5); and generalizing and justifying (P #7). All problems encourage the use of particular MPs including seeking to find and understand patterns (MP1), creating models (e.g., graphs or tables) for the mathematical situation in the problem (MP4), using logical arguments (MP7), and valuing and using symbolic representations of ideas in the problem (MP9).

Table 3. Different solution strategies to the given problems

Strategy	Description
Verbal	Using verbal reasoning to proceed from one factual statement to another in order to arrive at a solution.
Numerical	Using a systematic procedure to iteratively search for a solution. The systematic procedure must include guidance on how to iterate toward a solution.
Tabular	Using tabular arrays to enumerate the set of all valid solutions to equations to identify common elements (i.e., solution/s common to all equations).
Symbolic	Representing unknown quantities using variables; representing relationships between variables using equations; using algebraic approaches to solve the equations to determine the value/s of the unknown quantities.
Graphical	Representing unknown quantities using variables; representing relationships between variables using equations; plotting equations and using the plot to determine the value/s of the unknown quantities.

Data Analysis

Data for this paper come from a total of fourteen reflection forms–one for each problem (P #1 to P #7) from each teacher–which primarily include mathematical text (verbal, visual, and symbolic) (Dostal & Robinson, 2018). These data were analyzed by Hatisaru and Richardson. We used open coding with a deductive mode of inquiry (Khandkar, 2009), and codes were derived from research literature.

We analyzed each reflection form in two steps and focused on the responses to prompt #1; the responses from other prompts were only used to find supporting evidence for the findings derived from prompt #1. Extensive discussions were held in each step. First, we identified the solution strategies generated by the teachers to discover how they approached the given problems: verbal, numerical, tabular, graphical, and symbolic (see **Table 3**).

Next, we conducted an in-depth analysis of these solutions. We examined each of the solutions separately according to the MPs presented in **Table 1** and wrote memos:

> In solving the same problem, while one teacher uses verbal representation system (Mia), the other teacher uses a more visual/pictorial representation (X's model). Mia, however, suggests acting the problem with books. So, both activate MP4 (reflection form #3, P #3, Hatisaru).

> In solution 1, Levi demonstrates a misunderstanding of the difference between discrete and continuous sets. The method presented would work if you were able to list the elements of the set you are exhaustively exploring; however, it would not work for a continuum which cannot be listed. That is, you cannot apply this method to non-integers. This is another lack of MP2 not understanding structure, and MP3 (reflection form #6, P #6, Richardson).

We then compared the memos we wrote, and any disagreement was resolved by discussion. The data from Levi and Mia were analyzed separately to observe any possible similarities and/or differences in their responses. A practice was classified as lacking when a teacher's solution was assessed to be deficient based on the absence of that practice. There were numerous instances in which the teachers did not seize an opportunity to demonstrate a practice; however, this did not detract from their solution and so the practice was not classified as lacking (i.e., opportunity to demonstrate a practice does not equate to necessity). The following are comprehensive elaborations of the findings. Due to space, we describe the teachers' solutions but provide the actual solutions in the **Supplementary Information File** (SIF).

We have incorporated several strategies to ensure the credibility of the study and its findings, informed by Noble and Smith (2015). For example, we have provided a transparent and clear description of the research process from its methods, including data generation and analysis, to the reporting of findings. Data were analyzed by two research team members (Hatisaru and Richardson) who have extensive expertise both in the relevant content areas (mathematics and mathematics education) and in qualitative data analysis. These two team members had extensive discussions to understand the MPs themselves (presented in **Table 1**) and their manifestation in actual contexts (in teacher solutions to the problems) to assist them to uncover taken-for-granted assumptions. The written reflection forms allowed for repeated revisiting of the solutions to check MPs' occurrences and remain true to teachers' use of MPs in solving these problems. Finally, we have presented rich and thick descriptions of teachers' solutions, as well as all actual solutions (see SIF). Based on

Table 4. Summary of the teachers' solution strategies to problem #1 to problem #7

Problem #	*Levi*	*Mia*
P #1	Symbolic; graphical	Symbolic; verbal; tabular
P #2	Tabular; symbolic	Tabular; graphical; symbolic
P #3	Verbal/visual; symbolic	Verbal; symbolic; tabular; physical
P #4	Symbolic	Symbolic; visual; verbal/symbolic
P #5	Symbolic	Symbolic; visual/symbolic
P #6	Numerical; symbolic	Numerical; symbolic
P #7	Numerical; symbolic	Numerical; symbolic

Table 5. MPs in/visible in the teachers' solutions to the problems

Coverage	Problem #	Levi	Mia
Representational	P #1	MP4; **MP5**; **MP8**; MP9	MP4; **MP7**; **MP8**; MP9;
	P #2	**MP2**; MP4; **MP5**; **MP8**; MP9	MP4; **MP5**; **MP7**; **MP8**; MP9
	P #6	MP1; **MP2**; **MP3**; **MP7**; **MP8**; MP9	MP1; **MP8**; MP9
Rule-based	P #3	MP1; MP4; MP7; MP9	MP1; MP4
	P #4 and P #5	**MP2**; MP7; **MP8**	MP2; MP4; MP7
Generalizing and justifying	P #7	MP1; **MP7**; **MP8**; MP9	MP1; **MP2**; **MP5**; **MP7**; MP9

Note. MPs marked in **bold** lack in the teachers' respective work

these strategies, we believe the reader is in a good position to make judgements about the trustworthiness of the study and findings.

FINDINGS

The RQs explored in this study were:

RQ1: How do the teachers solve the given problems? What strategies do they generate?

RQ2: What mathematical practices are manifested in the teachers' solutions?

The following subsections give the strategies both teachers generated to solve the given problems (RQ1). Findings regarding the MPs manifested, or not, in these solutions (RQ2) are presented in the next subsections.

Strategies Generated by Teachers to Solve the Given Problems (RQ1)

When presented with problems requiring a numerical answer (e.g., P #4 and P #5) Levi and Mia were able to determine the solution. For problems requiring proof or justification (e.g., P #6 and P #7), they were able to demonstrate reasonable understanding, but they were not always able to provide complete solutions. This might be the result of having had limited exposure to using precise mathematical definitions and generalizing results beyond a specific example. **Table 4** summarizes the various strategies employed by Levi and Mia to solve the given problems. As seen in **Table 4**, Mia generated more varied strategies than Levi to solve the first five problems, whilst both used the same strategies to solve the last two problems.

MPs in Teacher Solutions to Given Problems (RQ2)

The MPs that were present, or lacked, in the teachers' mathematical solutions are given in **Table 5**. In the following subsections, we give comprehensive elaborations of these MPs, organized into three groups according to the mathematical content that the respective problems cover.

MPs in Solutions to Problems #1, #2, and #6

P #1 and P #2 are relatively standard in the sense that their solutions require the determination of a specific numerical solution, while P #6 is more advanced in that it requires a result to be demonstrated (see **Table 2**). These three problems were considered together as they are all based on the idea of simultaneous equations and can be approached using simultaneous equations.

MPs in solutions to P #1

Levi solved P #1 algebraically by devising simultaneous equations and created a graphical solution (see SIF, pp. 1–2, solution 1 and solution 2). However, the conceptual connection between algebraic and graphical solutions was not recognized. That is, the simultaneous equations in solution 1 were formulated to solve for the number of cows and the number of chickens, whilst the graphical approach in solution 2 involved plotting the number of legs against the number of chickens. The language used to set up the algebraic approach was imprecise (MP5). A precise definition would be to 'let A = the number of cows', rather than 'cows = A'. The absence of the use of precise definitions (MP5) and language (MP8) in creating mathematical text was also evident in the graphical solution.

Mia used guess and check approaches to solve P #1 (solution A and solution B; SIF, p. 1). Additionally, she solved the problem algebraically (solution C) and using a table (solution D). Solution A was interesting as the approach was not particularly intuitive and was not well explained. It left us questioning whether the limitations of the approach were recognized, and whether its success was simply a fluke. We thought that this was a case of Mia performing a sequence of steps that happened to deliver the correct solution, and then concluding that the sequence of steps was valid as a result. This reflected a lack in MP7 in relation to making logical arguments and/or a lack in MP1 in relation to seeking to understand a pattern or argument. The algebraic approach provided in solution C was outlined poorly. The definition of variables was not precise, the origin of the equations was not explained, and most algebraic steps were omitted (MP8). However, the opportunity to simplify the solution process by dividing equation $2a + 4b = 62$ by a common factor of 2 was noticed.

MPs in solutions to P #2

Levi generated a tabular solution to P #2 (solution A) and devised simultaneous equations (solution B) (see SIF, p. 3). He did not propose any form of graphical solution, despite having proposed a graphical approach to P #1. This observation suggested that not a clear connection between the similar underlying mathematical structure of P #1 and P #2 was drawn (MP2).

In solution A, how the equation was manipulated to give B was not outlined. That is, where $3A–5$ came from was not explained. These might indicate a lack in MP5 and/or MP8 in terms of the precise use of definitions and symbolic language when creating symbolic text. Solution B is incomplete, although it was possibly a minor issue. That is, the value of dice A was determined, but not the value of dice B. Also, the variables in either solution were not defined, whilst the meaning was clear by the context. Ideally, it could have been written 'Let A denote the number rolled on dice A and B …'.

Mia generated three solutions and proposed an additional solution to P #2 (SIF, p. 3). She found the value of die A and B by making a list of possibilities in a table (solution A), presented a graph (solution B), and devised two equations representing the conditions in the problem (solution C). Some lack of explanation or details were seen in these solutions, and this raised the question of whether it was recognized that the underlying structure of P #2 is essentially identical to P #1. The tabular solutions generated by both teachers (solution A in both) were effectively the same. However, the one generated by Mia was far weaker in terms of explanation and using a systematic or exhaustive approach. This was perhaps a lack in terms of MP7 with respect to conducting efficient logical arguments.

In her reflections on P #2, Mia recognized the graphical approach to the problem (solution B), which she did not have on her reflections on P #1. This graph is imprecise with no additional text (e.g., defining axes) or statement of the solution (MP5). It was unknown whether Mia became aware of that approach upon discussions in meeting #1 where P #1 was the focus, or whether she viewed the problems as being different.

In solution D, Mia proposed a guess and check approach that (in her words) "[is] similar to the table but more random", which is in opposition to MP1 with respect to invisible effort to understand a pattern and MP2 to understand a structure. To us, guess and check approaches only becomes mathematical if one uses some intelligence or information about the structure of the problem or equations to guide their guesses.

MPs in solutions to P #6

In solving P #6, both teachers recognized the potential to use simultaneous equations and proposed specific examples to demonstrate a solution procedure. However, the solutions were not generalized. It seemed neither teacher had a firm grasp of the difference between showing a general result and establishing a specific example of that result, because neither of them produced a solution that could be considered to have fully answered the question.

In his "solution 1–making a list", Levi gave a specific case and then asserted without proof that it could be generalized (see SIF, p. 7). In there, Levi also demonstrated a misunderstanding of difference between discrete and continuous sets (integers versus real numbers). The method presented works if one can list the elements of the set they are exhaustively exploring, but it does not work for a continuum which cannot be listed. That is, one cannot apply this method to non-integers. This might be due to a lack of MP2 regarding not understanding structure, and MP3 as there was an attempt to conjecture but the conjecture was incorrect. It is a pity that Levi stated he would use this solution in his teaching:

> Definitely the first one [solution 1]. It's accessible to everyone, even students who don't have a strong grasp of algebra. I feel like it makes intuitive sense, too (reflection form #6, P #6, prompt #3).

In "solution 2–algebra and substitution", Levi started with an unnecessary assumption (SIF, p. 7). Whilst the algebraic solution first looked promising, it ended abruptly because the solution was reverted to a specific example again. It seemed that Levi did not recognize that he was just solving simultaneous equations perhaps because the right-hand side values "X" and "Y", representing the sum and difference of two numbers, were not specific numbers. When prompted on whether this problem would be useful for the year levels they teach, or whether it resembles problems they might use, Levi wrote:

> It would be a very useful question as a lesson starter or warm up. It doesn't fit neatly into too many of the units we teach, at least not at first glance (reflection form #6, P #6, prompt #2).

This was further evidence for a possible lack in MB2 by the fact that Levi could not identify a fit for the problem within the units he teaches, when it would easily fit into a unit including simultaneous equations.

Mia did recognize that she could use simultaneous equations, despite not formulating a general approach. In solution A and solution B, Mia used a specific example to demonstrate a method: i.e., simultaneous equations by elimination and by substitution. In solution A, the word 'combine' was used rather than 'add', and this might have been a lack of MP8. In solution C (the guess and check approach), Mia sought to find patterns or regularity, although it was less structured than solution 1 of Levi (SIF, p. 7).

It is worth noting that, in meeting #6 where discussions were on P #6, both teachers wanted to know a complete algebraic solution to the problem. Also, they were curious about why the relationship in the problem worked. To that end, they both valued an understanding of why relationships make sense (MP6), although they were unable to demonstrate it in their solutions.

MPs in Solutions to Problems #3, #4, and #5

P #4 and P #5 required the teachers to solve linear equations in various forms. In these two problems, the equations are not related to a context. P #3, on the other hand, is a context-based problem, and although it can be formulated and solved algebraically, it can be more efficiently solved using iterative reasoning working backward from a final state. As all these problems include a common mathematical content (linear equations, see **Table 2**), we considered them together.

MPs in solutions to P #3

In his "Solution one: a visual method that starts from the end" (see SIF, p. 4), Levi created a model by using the symbol "X" to represent a book. In this solution, a logical argument was used (MP7) and communicated clearly, reflecting that the underlying pattern/structure of the problem was seen (MP1). In "Solution two: algebra" (SIF, p. 4), MP9 manifested itself. Here, although he did not 'solve' an equation as such, Levi derived a symbolic representation to easily facilitate the 'working backward' approach of "solution one". That is, he did derive a recursive algebraic expression and then used that equation to work backward to a solution. In addition to defining the variables used, Levi rearranged the equation so that the remaining number of books was the independent variable, and the number of books at the previous stage was the dependent variable. This allows the equation to be applied efficiently at each step.

Overall, Mia did have a grasp on how to effectively approach P #3, too. She generated four solutions (see SIF, p. 4). In her first two solutions MP1 was visible in terms of using patterns and regularity. Specifically, in solution A, Mia implemented the idea of working backward (like Levi) through the stages of distributing books and then checked the result at the end. Perhaps the only difference was that Levi used symbols (i.e., X's) to denote books which incorporates a visual/pictorial aspect. In solution B, like Levi, Mia derived a recursive algebraic expression and then used that equation to work backward to a solution. Here, Mia did not recognize the opportunity to rearrange her equation so that number of books remaining, *x*, was the independent variable. As a result, she had to solve for *x* at each stage rather than simply substituting into an equation to give her *x* (as Levi had done). Perhaps this reflected a lack of MP6 regarding seizing opportunities to explore relationships. However, based on the MPs observed in Mia's work in general, we decided that this incident might be accidental.

Mia's third approach: "Solution C: act it out with books" (MB4) would ultimately be the same idea as her solution A but would incorporate a physical aspect that mimics the X's in solution A of Levi. Mia "would hope that my [her] students would use strategy A, B, or C" in solving the problem (reflection form #3, prompt #4). We were surprised that solution C was considered a viable option as 22 books might not be easy to come by, although we guessed students could tear up a bit of paper with each piece representing a book. It appeared that Mia was unable to assess it as infeasible or inefficient. As opposed to the other three, Mia's fourth approach: "Solution D: guess and check" was not a particularly feasible approach without some rationale to direct guesses. We found it not mathematical in the way that it was presented, and it looked like 'randomly stabbing in the dark'.

MPs in solutions to P #4 and P #5

Levi's solutions to both P #4 and P #5 (see SIF, pp. 5–6) demonstrated MP7 while they were lacking in MP2 and MP8. We considered that MP7 was present in that performing algebraic operations requires the use of a logical argument. Also, as evidenced in Levi's comments to P #4a (not P #4b), he clearly understood the underlying reasoning behind the operations that he performed. It was interesting that Levi did not present a second solution to P #4b that would have aligned to solution 2 of P #4a where he expanded the bracket and then solved the equation. This might be either a sign that he did not invest a significant amount of time into the activity, or that he did not recognize common structural aspects of the two problems. In assessing whether P #4a and P #4b would be useful for his students and resemble problems he might use, Levi stated that:

> This would certainly be useful. We spend a lot of time working with students on opposite functions, working backwards, and showing correct algebraic working out. The second one (the longer one) would be the sort of question I'd use to demonstrate to students that this is not harder, just longer (reflection form #4, P #4, prompt #2).

The language used here such as "opposite functions" rather than 'inverse' is against the use of precise mathematical language (MP8).

The MPs observed in Levi's work on P #5 were consistent with the ones observed in P #4: demonstrating MP7 and lacking in MP8. Standard algebraic approaches in solving the given equations were included. There were two issues with using precise language–i.e., MP8. The first was the use of the term 'by inspection':

> $3x = 12$.
>
> Therefore, $x = 4$ (by inspection) (reflection form #5, P #5a).

Here, the term 'by inspection' is used to mean 'dividing both sides of the equation by 3'. In most instances Levi provided appropriate descriptions of his algebraic steps, so it was unknown why he made this choice on multiple occasions (P #5a, P #5b, and P #5c). The second was, when identifying the solution(s) he would desire from his students, Levi wrote:

> I would hope students would find that collecting "like terms" brackets is often a much quicker and more accurate way forward, but only where the option exists (reflection form #5, prompt #4).

The suggestion that one algebraic approach is “more accurate” than another, when both approaches are equally valid and accurate, is imprecise.

Mia created pictures or models for P #4a (not P #4b), mirroring the use of MP7, and MP4. In doing so she demonstrated MP2 within P #4a but did not consistently demonstrate that practice across P #4a and P #4b. She only presented one solution to P #4b, which might suggest that the problem was perceived to be fundamentally different to P #4a. As they are structurally similar, the cover up method Mia used in P #4a would have been an efficient way of solving the equation in P #4b that includes multiple steps. Solution C to P #4a was incorrect as it did not match the equation being solved because 4 times 4 is equal to 16 not 16*x*. Mia only yielded the correct answer by accident because that answer happened to be $x = 1$. That said, here, Mia demonstrated an attempt at exploring the structure of the problem from a different perspective. It is unknown whether the error was related to a lack of attention to detail or a lack of competence in interpreting the equation.

The same MPs were seen in solving the equations in P #5, as in P #4: MP7, MP4, and MP2. Mia demonstrated a greater appreciation of the structure of the problems based on the multiple approaches presented (see SIF, pp. 5–6). It is noteworthy that when creating mathematical text in general, and in their solutions to P #4 and P #5 in particular, neither teacher aligned = signs or used implication signs in presenting working out. We were unsure if that might be more about their inexperience with typing mathematics. Given that Mia presented her working out to P #5 in handwritten format, it was clearer that Mia did not adhere to the convention of aligning = signs or including implication signs, which we believe helps greatly in the clarity of presentation.

MPs in Solutions to P #7

In solving P #7a, both Levi and Mia used numerical examples to get a sense of what was going on before progressing to algebra. In solving P #7b and P #7c, they primarily used letters to express the general result (see SIF, p. 8). Both teachers got down to the essence of what they needed to show, although the structure of their solutions could be improved. For example, overall, Levi gave a complete solution to P #7a. The following concluding sentence could have been demonstrated algebraically:

> If the square of the middle term is n^2 and the product of the outer terms is n^2–1 then we see why the product is always one less than the square of the middle term (Levi, reflection form #7, P #7a).

Also, in this sentence the word ‘if’ was used inappropriately to refer to the truth of a known fact (i.e., square of n is n^2) which suggested a lack in terms of MP8. The structure of the solution to P #7b would have been improved if it had been started by defining x and then proceeding to the equation; that is, ‘define a variable before using it’. Perhaps this was a lack in MP7 relating to constructing logically sequenced arguments.

On a relevant note, all three problems themselves are vague in the sense that they clearly mean integers because ‘consecutive real numbers’ is meaningless, but the problems just refer to numbers. This vagueness was carried over by Levi in his solution to P #7c in that $2n$ is only an even number if n is an integer, but Levi did not make that statement. Solution P #7c is correct, although lacks the use of algebra:

> Let the lower number be ‘*n*’ and the upper number be ‘*n*+1’. Clearly if we add them together, we get ‘2*n*+1’, which by definition is an odd number (Levi, reflection form #7, P #7c).

While this sentence explaining the result is correct, it could have been written $n + (n + 1) = 2n + 1$ which is odd. This was perhaps a reflection of the use symbolic language competently where relevant (MP9).

Some similar undesired practices were observed in Mia’s solutions to P #7. In these solutions, for example, variables were not defined at all (MP5). Whilst Mia did better than Levi in using algebra to make her conclusion in P #7a, like Levi, she could have concluded the solution by writing the desired equation: $n^2 – (n – 1) \times (n + 1) = 1$. Mia presented two solutions (solution A and solution B) to P #7b which were structurally identical, and the only difference was superficial. The solution to P #7c was incomplete. Here, Mia generated a solution but did not nail it down, referring to the definition of an even number (MP7 and MP5).

Table 6. Summary of MPs evident or lacking in the teachers' solutions to P #1 to P #7

MPs	Levi	Mia
MP1: Seek to find and understand patterns.	√ √ √	√ √ √
MP2: Classify and characterize objects based on structure.	× × × ×	√ √ ×
MP3: Make and test conjectures about objects and structures.	×	
MP4: Create mental and/or physical models for, and examples and non-examples of, mathematical objects.	√ √ √	√ √ √ √ √
MP5: Value and use precise definitions of objects.	× ×	× ×
MP6: Value an understanding of why relationships make sense.	√	√
MP7: Value and use logical arguments and counterexamples as sources of conviction.	√ √ √ × ×	√ √ × × ×
MP8: Value precise language and have fine distinctions about language.	× × × × × ×	× × ×
MP9: Value and use symbolic representations of, and notation for, objects and ideas.	√ √ √ √ √	√ √ √ √

SUMMARY OF FINDINGS AND DISCUSSION

The teachers were able to solve the problems whose goal was to arrive at a numerical answer (e.g., P #4 and P #5) and were able to demonstrate a reasonable degree of understanding of the 'prove or justify' type problems (i.e., P#6 and P#7). Their failure to produce complete solutions to P#6, and to some questions within P#7, was likely a consequence of their limited knowledge or experience in using precise definitions and generalizing results (i.e., showing a general result as opposed to providing a supporting example). In agreement with Copur-Gencturk and Doleck (2021), when the teachers were able to generate valid strategies to solve a given problem, they arrived at the correct answer.

Table 6 summarizes the MPs that were evident, or not, in the teacher responses to the problems. Each tick represents an instance where the practice was demonstrated, while each cross represents an instance where the practice appeared lacking. Based on the findings, the MPs were divided into three groups: those that were consistently demonstrated; those that were demonstrated in routine problems but not in others; and those that were not demonstrated.

It is worth noting that MP3 and MP6 were not classified into any of the three groups. It was not necessary to demonstrate MP3 to answer any of the problems, so while none of Mia's solutions were assessed as demonstrating MP3, they also were not specifically lacking. The only instance in which Levi conjectured, his conjecture was incorrect–i.e., in P#6 he conjectured that a method based on enumeration of integer cases could be extended to the real number. None of the teacher solutions demonstrated (or lacked) in MP6. The ticks against MP6 in **Table 6** were based on the teacher discussions in the study group meeting #6 in which both teachers sought a detailed explanation of the problem and how and why the relationship worked.

Consistently Demonstrated MPs

Both teachers consistently demonstrated MP1, MP4, and MP9. It is fair to say that Mia created mental or concrete models for ideas in the problems more often than Levi, and Levi was inclined to use algebraic symbolism more often than Mia.

MP1 and MP4 are linked to seeking and establishing understanding of a problem, while MP9 is about representing problems mathematically. In some ways these could be considered the most fundamental of the MPs regarding making sense and representing mathematically, so it is reassuring that these practices were evident in both teachers' solutions. The slight difference revealed in the responses of Levi and Mia might be influenced by their teacher training (Blömeke & Delaney, 2012). That is, Mia has a primary education background that included some elective mathematics units, whereas Levi has a science major with mathematics being the minor subject where he studied calculus and algebra. It is likely that the use of physical models (e.g., the use of diagrams, counters, or blocks) as tools to assist students to explore and understand mathematics is more prevalent in the primary school context (Ball, 2000; Seaman & Szydlik, 2007) and in the associated university electives. It is probable that these tools and processes are more fundamental to Mia's notion of doing mathematics than the use of algebraic symbolism. Conversely, Levi's training would likely have focused far less on physical models and exploration, and more on the introduction of algebraic methods for representing and solving problems. However, this finding is tentative and needs to be further investigated.

MPs Manifested in Routine Problems but Less Present in Nonroutine Ones

MP2 and MP7 were evident in the teacher solutions to P #4 and P #5, which are routine problems, but generally not present in their solutions to nonroutine problems such as P #6 and P #7, as was also observed in the solutions of participating teachers in Hatisaru et al. (2022). Both MPs represent a higher level of sophistication, including the ability to argue logically, recognize and generalize structure, and to conjecture. This is unsurprising, as the use of logical arguments and recognition of structure in unfamiliar contexts is a more advanced skill than the application of logical arguments and recognition of structure in familiar or standard ones. Success in generalizing and justifying type tasks (Kilpatrick et al., 2001)–such as P #6 and P#7–requires proficiency not only in the use of algebra/symbolic representation, but also a comfort or preparedness to explore unfamiliar ideas and create mental models. That is because, as described extensively by Kilpatrick et al. (2001), these tasks include, for example, problem-solving, modelling, justifying, proving, or predicting skills, and in these tasks, all aspects of mathematical proficiency come together.

Interestingly, Levi came closer to producing complete solutions to P #6 and P #7, most likely because of his greater training and experience in algebraic representation and reasoning. Mia, on the other hand, seemed to undertake simple explorations of the problems but lacked the algebraic proficiency to represent and generalize her findings. This again suggests that the quality of teacher use of MPs is mediated by learning experiences and opportunities to learn during teacher training, and broad policy or curriculum contexts are also contributing factors (Blömeke & Delaney, 2012). To our knowledge, in the Australian school curriculum (ACARA, 2022), compared to representational and transformational activities of algebra, generalization and justification activities are less common in grades 7 and 8 (12–13 years old)–the years that Levi and Mia usually teach. Possibly neither teacher explores this type of algebra activities themselves nor commonly uses it in their instruction. The teachers' use of MPs observed in their responses to the generalization and justification type problems in this sense might reflect what they enact less in their classes.

Consistently Lacked MPs

MP5 and MP8 were not evident in any of the teacher solutions. Both of these MPs relate to precision in language and definitions in creating and/or communicating textual, symbolic, visual, or graphical type of mathematical text (Dostal & Robinson, 2018). Broadly, we concur with Matsuura et al. (2013) that whether a teacher uses precise and clear mathematical language might depend on the classroom context. Within the context of this study, the lack of MP5 and MP8 perhaps may reflect a more casual or intuitive and less rigorous presentation of mathematics, both in university mathematics courses where teachers are trained, and subsequently in secondary school mathematics classrooms. Nevertheless, as Ball et al. (2005) stated, we believe that: "Mathematics requires careful reasoning about precisely defined objects and concepts" (p. 1055). To that end, we join Dostal and Robinson (2018) who highlighted the importance of fluency in mathematical text and language in enhancing the ability of a student (or teacher) to engage in mathematical thinking. MPs in general are rarely made explicit during teacher training as a facet of mathematical knowledge (Bauer & Kuennen, 2016). However, it is desirable that teachers of mathematics receive training in reading, interpreting, utilizing, and creating mathematical language, as well as in other MPs.

CONCLUSION

In this study, we analyzed solutions generated by two secondary mathematics teachers to a set of algebraic problems with the aim of determining the presence of and prevalence of several MPs used by mathematicians. Consistent with past research, these two teachers were largely able to compute numerical answers and to provide reasonable justifications in support of their answers to the algebraic problems–although they were challenged to provide complete solutions to some of the problems. In terms of their use of MPs, our results indicate that there were a set of practices that were consistently used by the two teachers, related to finding and understanding patterns, using mental and physical models, and using symbolic representations. But for other practices, these two teachers either did not take advantage of opportunities to use the practices (e.g., using precise definitions; using precise language) or the problems did not require the use of the practices (e.g., making and testing conjectures). These findings contribute to our understanding of the ways that secondary teachers can and should know mathematics, particularly in pointing our attention to how teachers can know and do mathematics as mathematicians do and inviting us to consider how teacher training experiences and curricula can foster the development of teachers' use of MPs used by mathematicians.

Limitations and Future Directions

We aimed to understand the extent to which secondary mathematics teachers use the practices of doing mathematics and have observed how the teachers use, or not, core practices of mathematicians in their mathematical work. Whilst the findings are limited to the solutions generated by two teachers, they contain valuable insight into the use of mathematical practices (MPs) among teachers of mathematics. As such, the study also opens new research avenues for research on the use of MPs in the teaching and learning of mathematics.

First, this study is based on teachers' written work to a set of problems. We need to understand how teachers activate these practices in their teaching, for example, in their lesson planning, instructions, and task design, sequencing and implementation. More importantly, as also suggested by Matsuura et al. (2013), we need to investigate the complex connections between the use of MPs in teaching and student learning. Broadly, a teacher's level of the use of these practices can inform the strategies they use to approach mathematical problems and consequently may impact their teaching, and the learning outcomes of their students (Seaman & Szydlik, 2007). Second, while analyzing the associations between the use of MPs and performance in problem solving was beyond the scope of this paper, it seems that the absence of specific MPs does not limit one in 'arriving at a correct answer'. As Tran and Munro (2019) asked: "Does possession, or absence, of these [practices] influence their [one's] ability to solve the problems?" To us, this is an emergent area. An important consideration going forward is the design of mathematical tasks which are specifically designed to elicit MPs and less about obtaining a correct answer. Such tasks may facilitate an enhanced assessment of teachers' level of the use of MPs. Third, in this investigation, we have considered MPs as part of teacher mathematical knowledge needed for teaching; however, examining the participating teachers' mathematical knowledge was beyond the scope of the investigation. We are therefore unable to make judgements about if and how teachers' performance in using MPs may be mediated by their mathematical knowledge needed for teaching. We hope that researchers in this field will take this opportunity.

Finally, in accord with Blömeke and Delaney (2012), our data show that teachers' relevant performance could have been impacted by their learning experiences during teacher training. This finding, nevertheless, is tentative and needs to be explored further. Research studies including larger groups of teachers who have different teacher training are warranted to observe this potential impact.

Author contributions: VH: generating the data, conceptualizing the paper, data analysis, writing the paper, editing; **SR:** data analysis, writing the paper, proofreading; **JRS:** contributing to the writing and critically reviewing the paper. All authors approved the final version of the article.

Funding: The authors received no financial support for the research and/or authorship of this article.

Acknowledgments: The authors would like to thank the University of Tasmania which approved the research, the Mathematical Association of Tasmania for supporting the implementation of, and the teachers who participated in, the study. Any opinions, findings, and conclusion or recommendations expressed in this paper are those of the authors and do not necessarily reflect the view of these institutions.

Ethics declaration: This study was approved by the University of Tasmania Human Research Ethics Committee with the Project ID: 23451. Informed consent was obtained from all participants.

Declaration of interest: The authors declared no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

REFERENCES

ACARA. (2022). *Australian curriculum: Mathematics (F-10 version 9). About the learning area*. Australian Curriculum, Assessment and Reporting Authority.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241–247. https://doi.org/10.1177/0022487100051003013

Ball, D., Ferrini-Mundy, J., Kilpatrick, J., Milgram, J., Schmid, W., & Schaar, R. (2005). Reaching for common ground in K-12 mathematics education. *Notices of the American Mathematical Society, 52*(9), 1055–1058.

Bauer, T., & Kuennen, E. (2016). *Building and measuring mathematical sophistication in pre-service mathematics teachers*. https://www.mathematik.uni-marburg.de/~tbauer/2016-Building-and-measuring-mathematical-sophistication.pdf

Blömeke, S., & Delaney, S. (2012). Assessment of teacher knowledge across countries: A review of the state of research. *ZDM Mathematics Education, 44*, 223–247. https://doi.org/10.1007/s11858-012-0429-7

Boesen, J., Lithner, J., & Palm, T. (2018). Assessing mathematical competencies: An analysis of Swedish national mathematics tests. *Scandinavian Journal of Educational Research, 62*(1), 109–124. https://doi.org/10.1080/00313831.2016.1212256

Copur-Gencturk, Y., & Doleck, T. (2021). Strategic competence for multistep fraction word problems: An overlooked aspect of mathematical knowledge for teaching. *Educational Studies in Mathematics, 107*, 49–70. https://doi.org/10.1007/s10649-021-10028-1

Crowe, S., Cresswell, K., Robertson, A., Huby, G., Avery, A., & Sheikh, A. (2011). The case study approach. *BMC Medical Research Methodology, 11*, Article 100. https://doi.org/10.1186/1471-2288-11-100

Cuoco, A., Goldenberg, E. P., & Mark, J. (2010). Contemporary curriculum issues: Organizing a curriculum around mathematical habits of mind. *Mathematics Teacher, 103*(9), 682–688. https://doi.org/10.5951/MT.103.9.0682

Dostal, H. M., & Robinson, R. (2018). Doing mathematics with purpose: Mathematical text types. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 91*(1), 21–28. https://doi.org/10.1080/00098655.2017.1357409

Glover, E.D. (2019). *An exploratory study of mathematics disciplinary practices in advanced mathematics* [Doctoral dissertation, Oregon State University].

Hatisaru, V. (2024). Non-specialist secondary mathematics teachers learning in study groups by engaging with activities of algebra. *Journal on Mathematics Education, 15*(1), 131–150. http://doi.org/10.22342/jme.v15i1.pp131-150

Hatisaru, V., Oates, G., & Chick, H. (2022). Developing proficiency with teaching algebra in teacher working groups: Understanding the needs. In N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 250–257). MERGA.

Hatisaru, V., & Erbaş, A. K. (2017). Mathematical knowledge for teaching and student learning outcomes. *International Journal of Science and Mathematics Education, 15*(4), 703–722. https://doi.org/10.1007/s10763-015-9707-5

Hill, H. C., & Chin, M. (2018). Connections between teachers' knowledge of students, instruction, and achievement outcomes. *American Educational Research Journal, 55*(5), 1076–1112. https://doi.org/10.3102/0002831218769614

Ito-Hino, K. (1995). Students' reasoning and mathematical connections in the Japanese classroom. In P. House, & A. F. Coxford (Eds.), *Connecting mathematics across the curriculum* (pp. 233–245). National Council of Teachers of Mathematics.

Khandkar, S. H. (2009). *Open coding*. http://pages.cpsc.ucalgary.ca/~saul/wiki/uploads/CPSC681/opencoding.pdf

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 390–419). Macmillan Publishing Co, Inc.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. The National Academy Press. https://doi.org/10.17226/9822

Levasseur, K., & Cuoco, A. (2003). Mathematical habits of mind. In H. L. Schoen (Ed.), *Teaching mathematics through problem solving: Grade 6-12* (pp. 23–37). National Council of Teachers of Mathematics.

Lim, K. H. (2008). Advancing pre-service teachers' mathematical sophistication via mathematical tasks. *EduMath, 27*, 2–24.

Lim, K. H. (2013). *A collection of lists of mathematical habits of mind*. http://math.utep.edu/faculty/kienlim/HoM_Collection.pdf

Martín-Molina, V., González-Regaña, A. J., & Gavilán-Izquierdo, J. M. (2018). Researching how professional mathematicians construct new mathematical definitions: A case study. *International Journal of Mathematical Education in Science and Technology, 49*(7), 1069–1082. https://doi.org/10.1080/0020739X.2018.1426795

Matsuura, R., Sword, S., Piecham, M. B., Stevens, G., & Cuoco, A. (2013). Mathematical habits of mind for teaching: Using language in algebra classrooms. *Mathematics Enthusiast, 10*, 735–776. https://doi.org/10.54870/1551-3440.1285

Moschkovich, J. N. (2013). Issues regarding the concept of mathematical practices. In Y. Li, & J. N. Moschkovich (Eds.), *Proficiency and beliefs in learning and teaching mathematics. Mathematics teaching and learning* (pp. 257–275). Sense Publishers. https://doi.org/10.1007/978-94-6209-299-0_15

Musser, G. L., Burger, F. W., & Peterson, B. E. (2008). *Mathematics for elementary teachers: A contemporary approach* (10th ed.). Wiley.

NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. National Council of Teachers of Mathematics.

NGA Center & CCSSO. (2010). *Common core state standards for mathematics*. National Governors Association Center for Best Practices & Council of Chief State School Officers.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies of Mathematics, 102*, 9–28. https://doi.org/10.1007/s10649-019-09903-9

Noble H., & Smith J. (2015). Issues of validity and reliability in qualitative research. *Evidence-Based Nursing, 18*(2), 34–35. https://doi.org/10.1136/eb-2015-102054

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education, 10*, 167–182. https://doi.org/10.1007/s10857-007-9033-0

Star, J. R., & Seifert, C. (2006). The development of flexibility in equation solving. *Contemporary Educational Psychology, 31*(3), 280–300. https://doi.org/10.1016/j.cedpsych.2005.08.001

Szydlik, J., Kuennen, E., & Seaman C. (2009). *Development of an instrument to measure mathematical sophistication*. In *Proceedings of the 12th Conference of the MAA's Special Interest Group on Research in Undergraduate Mathematics Education*.

Tchoshanov, M., Cruz, M. D., Huereca, K., Shakirova, K., Shakirova, L., & Ibragimova, E. N. (2017). Examination of lower secondary mathematics teachers' content knowledge and its connection to students' performance. *International Journal of Science and Mathematics Education, 15*, 683–702. https://doi.org/10.1007/s10763-015-9703-9

Thomson, S., De Bortoli, L., Underwood, C., & Schmid, M. (2019). *PISA 2018: Reporting Australia's results. Volume I student performance*. Australian Council for Educational Research.

Tran, D., & Munro, D. (2019). Operationalize mathematical sophistication in a collaborative problem-solving: A conceptual paper. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 723–730). MERGA.

Tripathi, N. P. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in the Middle School, 13*(8), 438–445. https://doi.org/10.5951/MTMS.13.8.0438

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematical item demand: A work in progress. In K. Stacey, & R. Turner (Eds.), *Assessing mathematical literacy* (pp. 85–115). Springer. https://doi.org/10.1007/978-3-319-10121-7_4