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An alternative triangle area strategy

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Abstract
We explore the mathematics engagement of a group ofmathematics coaches, working in k-12 mathematics education. The incenter of a triangle is used to derive an alternative formula for the area of a triangle inspired by Usiskin, Peressini, Marhisotto, and Stanley (2002).

Keywords: mathematics education, technology, triangle incenter

Introduction

For the 2013-2014 academic year, monthly, daylong professional development sessions were conducted for a group of mathematics coaches. These coaches worked in elementary, middle, and high school classrooms. A theme throughout the year was the use of GeoGebra as a problem solving, exploratory tool, supported by the Common Core Mathematical Practices (2010). At the time of this exploration, the coaches had experiences using GeoGebra, so while they may not have been "expert" they were not neophytes in using the software. To motivate the exploration, coaches were each given a piece of Patty paper that had an arbitrary triangle drawn on it. Figure 1 shows an example of one of the triangles; coaches also had examples of scalene and obtuse triangles (recognizing that somescalene triangles are obtuse). The intent was for the coaches to fold the paper to "find" angle bisectors, and ultimately the point of concurrency, the incenter of the three angle bisectors. Readers may recognize the generalization as $\frac{1}{2}ap$ where a represents the apothem and p is the perimeter.

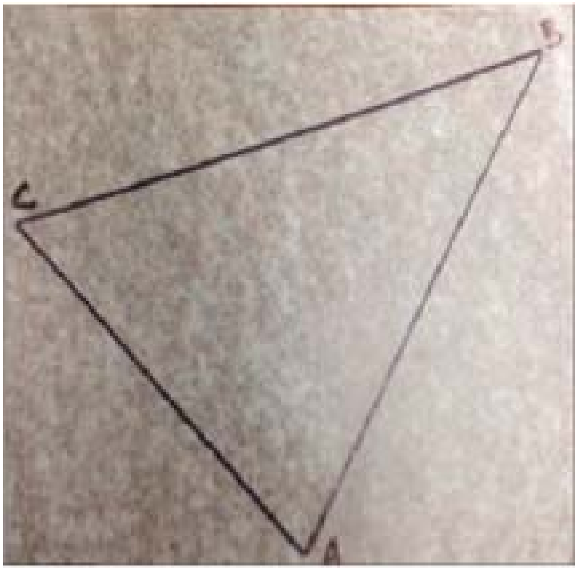

Figure 1 Patty paper triangle example, not intended to be isosceles

Exploration

By folding side $\overline{AB}$ onto side $\overline{AC}$, the wax surface of the Patty paper reveals the angle bisector of angle *A*. Our focus was solely upon the coaches repeating this folding at each vertex to reveal their triangle's incenter, (see Figure 2), occurring at the intersection.[insert figure2]

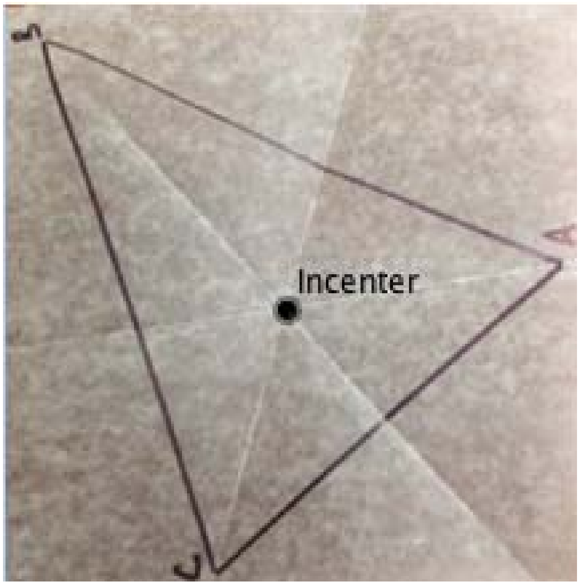

Figure 2 Triangle incenter revealed on Patty paper

This was not an expected outcome for any of the coaches. While the exploration was focused upon the alternative area formula for a triangle, we realized that the coaches were expanding their thinking with this brief introduction to one triangle center. We did other folds on their triangle, for example side bisectors, but we focus this writing on the incenter. Opening up a new mathematics experience for these coaches also revealed the need for language precision, as these coaches were learning new mathematical vocabulary with respect to triangle centers. With the new term of incenter, a coach used her phone to search this new vocabulary on the internet. She shared that the 'incenter was the center of circle that could be inscribed in the triangle.' We used this revelation to segue into more explorations of the incenter through GeoGebra.

Engaging GeoGebra

We engaged the use of GeoGebra, with a purposely-different approach. With paper folding the coaches were able to find the incenter, with GeoGebra we asked the coaches to begin with a circle. While this approach was “backward” for the coaches, we wanted them to see that the center of their circle was indeed the point of concurrency of the three angle bisectors of the triangle they circumscribed about their circle. Our strategy with GeoGebra had the coaches do the following:

1) Construct a circle;
2) Place three points on the circumference of their circle;
3) Use the built-in tangent tool (an opportunity to use a new feature) to construct three tangent lines to the points in step 2 (see Figure 3).

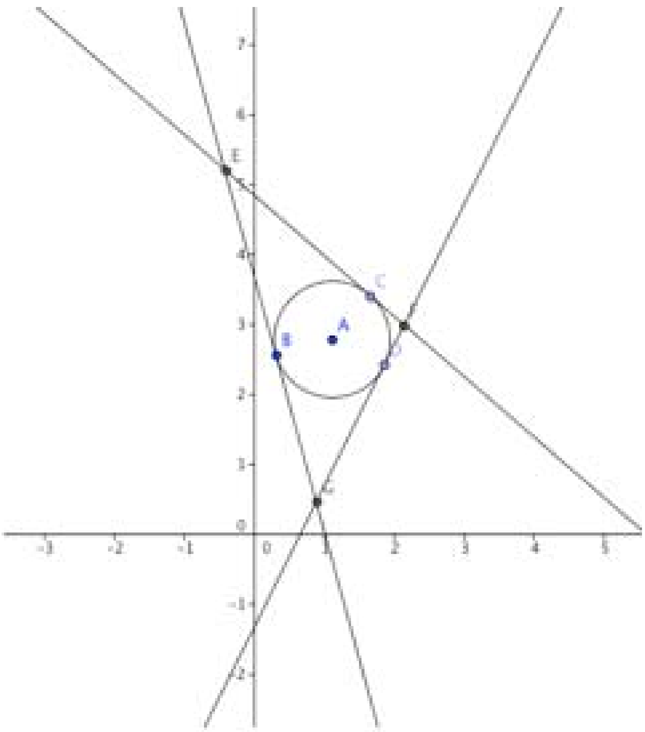

Figure 3

With the inscribed circle visually obvious, the coaches then used the built-in angle bisector tool to construct the three angle bisectors for triangle *EFG*(see Figure 4).

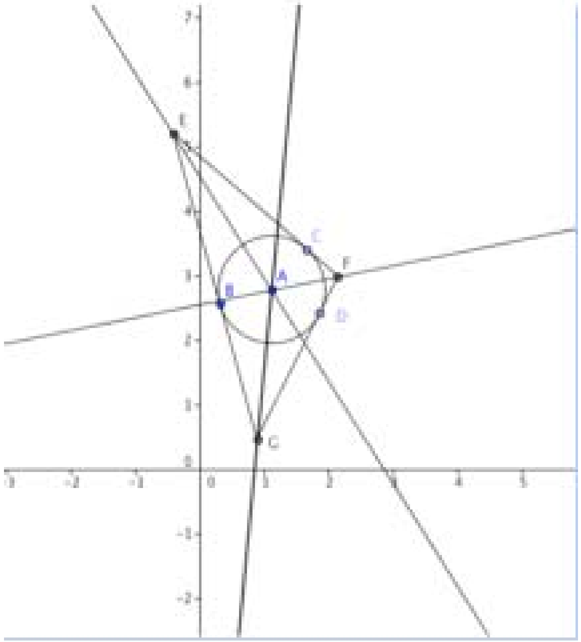

Figure 4 Angle bisectors concurrent at circle center

Triangle Area

As leaders of the professional development session, we knew with a little more effort there was enough information to find the area of triangle *EFG*, but as Driscoll (1999) reminds us, we cannot always "expect" learners to make mathematical connections. Initially the coaches wanted "numbers," that is they needed the dimensions of the triangle. We, the leaders, missed an opportunity by not exploring further the coaches' potential engagement with specific cases generated by the use of the built-in measurement tool. If we had listened to their strategies and how these strategies could have been used toward derivation of a generalization, further mathematical connections may have been revealed. Looking back we conjecture that these coaches use of specific cases could have yielded insights into how they determined altitudes so they could use the familiar $\frac{1}{2}bh$. Determining the height of the triangle was not obvious to the coaches, but they persisted.

We, the leaders of the session, anticipated that the coaches would decompose triangle *EFG* into three triangles, namely *EAF*, *GAF*, and *GAE*. (This anticipation was not completely correct, as once the altitudes were determined, say *AC*, one of the coaches decomposed our three triangles into six triangles. Triangle *EAF* became triangles *FAC* and *EAC*, triangle *GAE* became triangles *GAB* and *EAB*, and triangle *GAF* became triangles *GAD* and *FAD*.)

After discussion and quiet individual explorations, one of the coaches recalled that 'a radius is perpendicular to a point of tangency.' With this conjecture, coaches constructed radii and measured angles between the radius and the point of tangency, (see Figure 5). [insert figure 5]

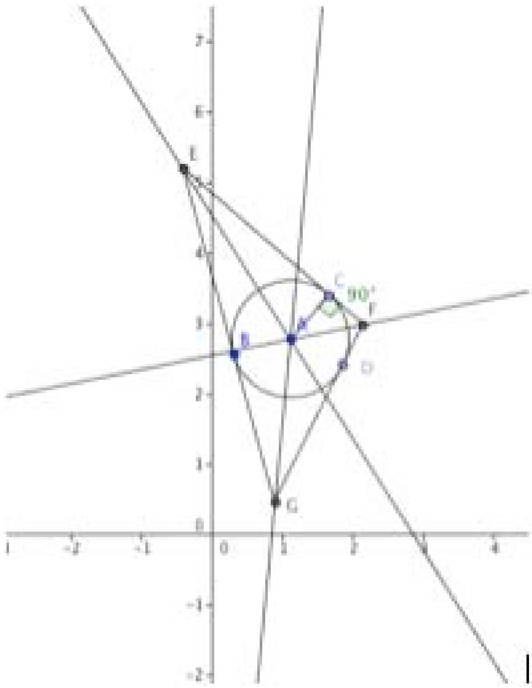

Figure 5 Checking angle measures between radius and point of tangency

Coaches checked each of the remaining altitudes, *AD* and *AB* and found the angle relationship held. Finding these three heights, led the coaches to use their familiar $\frac{1}{2}bh$ to find the three smaller triangles areas. As mentioned above, one coach found the area of the six smaller triangles she had decomposed triangle *EFG* into. Using this information, representing the area of triangle EFG as the sum of the smaller triangles resulted in:

$$\begin{aligned} A_{\Delta EFG} &= A_{\Delta EAF} + A_{\Delta GAF} + A_{\Delta GAE} \\ &= \frac{1}{2}\left(m\overline{AC}\right)\left(m\overline{EF}\right) + \frac{1}{2}\left(m\overline{AD}\right)\left(m\overline{GF}\right) + \frac{1}{2}\left(m\overline{AB}\right)\left(m\overline{EG}\right) \end{aligned}$$

Initially the coaches were satisfied with this result, but when asked if there were any common 'factors,' a coach recognized that each area had the radius measure in common. This recognition led to

$$\begin{aligned} A_{\Delta EFG} &= \frac{1}{2}\left(m\overline{AC}\right)\left(m\overline{EF}\right) + \frac{1}{2}\left(m\overline{AD}\right)\left(m\overline{GF}\right) + \frac{1}{2}\left(m\overline{AB}\right)\left(m\overline{EG}\right) \\ &= \frac{1}{2}r\left(m\overline{EF}\right) + \frac{1}{2}r\left(m\overline{GF}\right) + \frac{1}{2}r\left(m\overline{EG}\right) \\ &= \frac{1}{2}r\left(\left(m\overline{EF}\right) + \left(m\overline{GF}\right) + \left(m\overline{EG}\right)\right) \end{aligned}$$

These coaches did not "see" that $\left(m\overline{EF}\right) + \left(m\overline{GF}\right) + \left(m\overline{EG}\right)$ was the perimeter of the triangle. Once we traced these three sides on the image projected to a whiteboard, a coach recognized this sum as the perimeter. Now, collectively, we arrived at a generalization for the area of a triangle that was new knowledge for this group of coaches, $A = \frac{1}{2}rp$, where *r* represents radius of the inscribed circle and *p* represents perimeter. While the coaches were enjoying their new formula for the area of a triangle, a new question was posed, “Can this exploration be extended beyond triangles?”

Extending the Problem

We extended the above to explore finding the area of a pentagon. Modifying the GeoGebra triangle instructions from the above description, we asked the coaches to do the following:

1) Construct a circle:
2) Place five points on the circumference and;
3) Construct tangent lines at these five points (see Figure 6). (For illustration purposes we cleaned up the five tangent lines in Figure 6.)

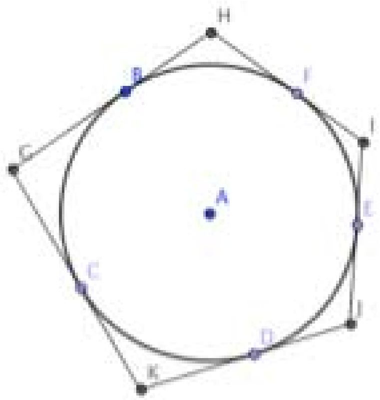

Figure 6 Pentagon with inscribed circle

As this group of coaches was attending to this problem, they began considering a ten-triangle decomposition, contrary to our anticipation of a five-triangle decomposition. This was another reminder that our learners’ views sometimes differ from our own. The ten-triangle decomposition were all right triangles, for example triangle *HAI* was decomposed into right triangle *HAF* and right triangle *IAF*, (see Figure 7).

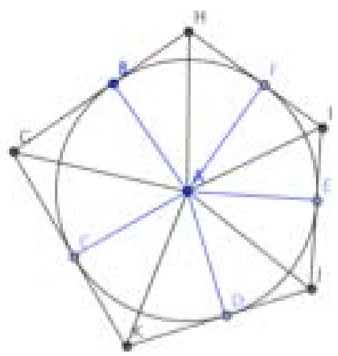

Figure 7Ten triangle decomposition

Using the coaches’ view of ten triangles, the area of the pentagon was found to be:

$$A=\frac{1}{2}\left(m\overline{AF}\right)\left(m\overline{FH}\right)+\frac{1}{2}\left(m\overline{AF}\right)\left(m\overline{FI}\right)+\frac{1}{2}\left(m\overline{AE}\right)\left(m\overline{EI}\right)+\frac{1}{2}\left(m\overline{AE}\right)\left(m\overline{EJ}\right)+\frac{1}{2}\left(m\overline{AD}\right)\left(m\overline{DJ}\right)+$$
$$\frac{1}{2}\left(m\overline{AD}\right)\left(m\overline{DK}\right)+\frac{1}{2}\left(m\overline{AC}\right)\left(m\overline{KC}\right)+\frac{1}{2}\left(m\overline{AC}\right)\left(m\overline{CG}\right)+\frac{1}{2}\left(m\overline{AB}\right)\left(m\overline{BG}\right)+\frac{1}{2}\left(m\overline{AB}\right)\left(m\overline{BH}\right)$$

Identifying the radii and factoring out common terms, the above reduced to:

$$A=\frac{1}{2}r\left(m\overline{FH}\right)+\frac{1}{2}r\left(m\overline{FI}\right)+\frac{1}{2}r\left(m\overline{EI}\right)+\frac{1}{2}r\left(m\overline{DJ}\right)+$$
$$\frac{1}{2}r\left(m\overline{DK}\right)+\frac{1}{2}r\left(m\overline{KC}\right)+\frac{1}{2}r\left(m\overline{CG}\right)+\frac{1}{2}r\left(m\overline{BG}\right)+\frac{1}{2}r\left(m\overline{BH}\right)$$
$$=\frac{1}{2}r\left\{\left(m\overline{FH}\right)+\left(m\overline{FI}\right)+\left(m\overline{EI}\right)+\left(m\overline{DJ}\right)+\left(m\overline{DK}\right)+\left(m\overline{KC}\right)+\left(m\overline{CG}\right)+\left(m\overline{BG}\right)+\left(m\overline{BH}\right)\right\}$$
$$=\frac{1}{2}rp$$

There is no loss of generalization decomposing to ten triangles or five triangles.

Conclusion

The mathematical practices (CCSM, 2010) are just as valuable for our inservice mathematics educators as they are for our preservice mathematics when we consider mathematical engagement. We agree with Driscoll (1999) about the importance of generalizations in mathematics. . Mason (1998) acknowledges the importance of mathematics teachers engaging in the content and opportunities such as this offer educator’s intellectual engagement in the content..These coaches were asking questions about this generalization and if it would work with other polygons, on the surface it seemed reasonable, but each coach was encouraged to determine if the results were the same for a polygon of their choosing. We chose GeoGebra as we feel it is an appropriate tool for exploring mathematical connections and mathematical conjectures. This group of mathematics coaches, we infer, left this session with mathematical connections that previously did not exist. They experienced new content, expanded their mathematical vocabulary, used technology as a problem-solving tool, and gained different ideas of how they might engage their learners. We believe their mathematical conceptual knowledge was enhanced, as was their mathematical procedural knowledge, and their abilities using GeoGebra. We engaged these coaches as learners, attempting to model how we hope they engage their teachers and students in their respective schools.

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